Jointly distributed Random variables

Multivariate distributions

Marginal and Conditional distributions

<u>Marginal distributions for the Bivariate Normal distribution</u>

Recall the definition of marginal distributions for continuous random variables:

$$f_1\left(x_1\right) = \int_{-\infty}^{\infty} f\left(x_1, x_2\right) dx_2 \quad \text{and} \quad f_2\left(x_2\right) = \int_{-\infty}^{\infty} f\left(x_1, x_2\right) dx_1$$

It can be shown that in the case of the bivariate normal distribution the marginal distribution of x_i is Normal with mean μ_i and standard deviation σ_i.

Proof:

The marginal distributions of x_2 is

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

$$= \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}Q(x_1,x_2)}\, dx_1$$

where

$$Q(x_1, x_2) = \frac{\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}}{1-\rho^2}$$

Now:

$$Q(x_1, x_2) = \frac{\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}}{1-\rho^2}$$

$$= \left(\frac{x_1-a}{b}\right)^2 + c = \frac{x_1^2}{b^2} - 2\frac{a}{b^2}x_1 + \frac{a^2}{b^2} + c$$

$$= \frac{x_1^2}{\sigma_1^2\left(1-\rho^2\right)} - 2\left[\frac{\mu_1}{\sigma_1^2\left(1-\rho^2\right)} + \rho\frac{x_2-\mu_2}{\sigma_2\sigma_1\left(1-\rho^2\right)}\right]x_1$$

$$\frac{\mu_1^2}{\sigma_1^2\left(1-\rho^2\right)} + 2\rho\frac{\left(x_2-\mu_2\right)}{\sigma_2\sigma_1\left(1-\rho^2\right)}\mu_1 + \frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2\left(1-\rho^2\right)}$$

Hence $b^2 = \sigma_1^2\left(1-\rho^2\right)$ or $b = \sigma_1\sqrt{1-\rho^2}$

Also
$$\frac{a}{b^2} = \frac{\mu_1}{\sigma_1^2\left(1-\rho^2\right)} + \rho\frac{x_2-\mu_2}{\sigma_2\sigma_1\left(1-\rho^2\right)}$$
$$= \frac{1}{\sigma_1^2\left(1-\rho^2\right)}\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(x_2-\mu\right)\right]$$

and
$$a = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(x_2-\mu\right)$$

Finally

$$\frac{a^2}{b^2}+c=\frac{\mu_1^2}{\sigma_1^2\left(1-\rho^2\right)}+2\rho\frac{\left(x_2-\mu_2\right)}{\sigma_2\sigma_1\left(1-\rho^2\right)}\mu_1+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2\left(1-\rho^2\right)}$$

$$c=\frac{\mu_1^2}{\sigma_1^2\left(1-\rho^2\right)}+2\rho\frac{\left(x_2-\mu_2\right)}{\sigma_2\sigma_1\left(1-\rho^2\right)}\mu_1+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2\left(1-\rho^2\right)}-\frac{a^2}{b^2}$$

$$=\frac{\mu_1^2}{\sigma_1^2\left(1-\rho^2\right)}+2\rho\frac{\left(x_2-\mu_2\right)}{\sigma_2\sigma_1\left(1-\rho^2\right)}\mu_1+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2\left(1-\rho^2\right)}$$

$$-\frac{\left[\mu_1+\rho\frac{\sigma_1}{\sigma_2}\left(x_2-\mu\right)\right]^2}{\sigma_1^2\left(1-\rho^2\right)}$$

and

$$c = \frac{1}{\sigma_1^2\left(1-\rho^2\right)}\left[\mu_1^2 + 2\rho\frac{\sigma_1}{\sigma_2}\mu_1\left(x_2-\mu_2\right) + \frac{\sigma_1^2}{\sigma_2^2}\left(x_2-\mu_2\right)^2 - \left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(x_2-\mu_2\right)\right]^2\right]$$

$$= \frac{1}{\sigma_1^2\left(1-\rho^2\right)}\left[\frac{\sigma_1^2}{\sigma_2^2}\left(1-\rho^2\right)\left(x_2-\mu_2\right)^2\right]$$

$$= \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2$$

Summarizing

$$Q\left(x_1, x_2\right) = \frac{\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}}{1-\rho^2}$$

$$= \left(\frac{x_1-a}{b}\right)^2 + c$$

where $b = \sigma_1\sqrt{1-\rho^2}$

$$a = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(x_2-\mu\right)$$

and $c = \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2$

Thus $$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

The picture can't be displayed.

$$= \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}Q(x_1, x_2)}\, dx_1$$

$$= \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\left[\left(\frac{x_1-a}{b}\right)^2 + c\right]}\, dx_1$$

$$= \frac{\sqrt{2\pi}\, b e^{-c/2}}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, b} \mathrm{e}^{-\frac{1}{2}\left(\frac{x_1-a}{b}\right)^2}\, dx_1$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

Thus the marginal distribution of x_2 is Normal with mean μ_2 and standard deviation σ_2.

Similarly the marginal distribution of x_1 is Normal with mean μ_1 and standard deviation σ_1.

Conditional distributions for the Bivariate Normal distribution

Recall the definition of conditional distributions for continuous random variables:

$$f_{1|2}\left(x_1 \middle| x_2\right) = \frac{f\left(x_1, x_2\right)}{f_2\left(x_2\right)} \quad \text{and} \quad f_{2|1}\left(x_2 \middle| x_1\right) = \frac{f\left(x_1, x_2\right)}{f_1\left(x_1\right)}$$

It can be shown that in the case of the bivariate normal distribution the conditional distribution of x_i given x_j is Normal with:

mean $\mu_{i|j} = \mu_i + \rho \dfrac{\sigma_i}{\sigma_j}\left(x_j - \mu_j\right)$ and

standard deviation $\sigma_{i|j} = \sigma_i \sqrt{1-\rho^2}$

Proof

$$f_{2|1}\left(x_2 \middle| x_1\right) = \frac{f\left(x_1, x_2\right)}{f_1\left(x_1\right)}$$

$$= \frac{\dfrac{e^{-\frac{1}{2}Q(x_1,x_2)}}{(2\pi)\sigma_1\sigma_2\sqrt{1-\rho^2}}}{\dfrac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}}$$

$$= \frac{e^{-\frac{1}{2}Q(x_1,x_2)-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}} = \frac{e^{-\frac{1}{2}\left[\left(\frac{x_1-a}{b}\right)^2+c\right]-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}$$

where $$b = \sigma_1 \sqrt{1 - \rho^2}$$

$$a = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}\left(x_2 - \mu\right)$$

and $$c = \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2$$

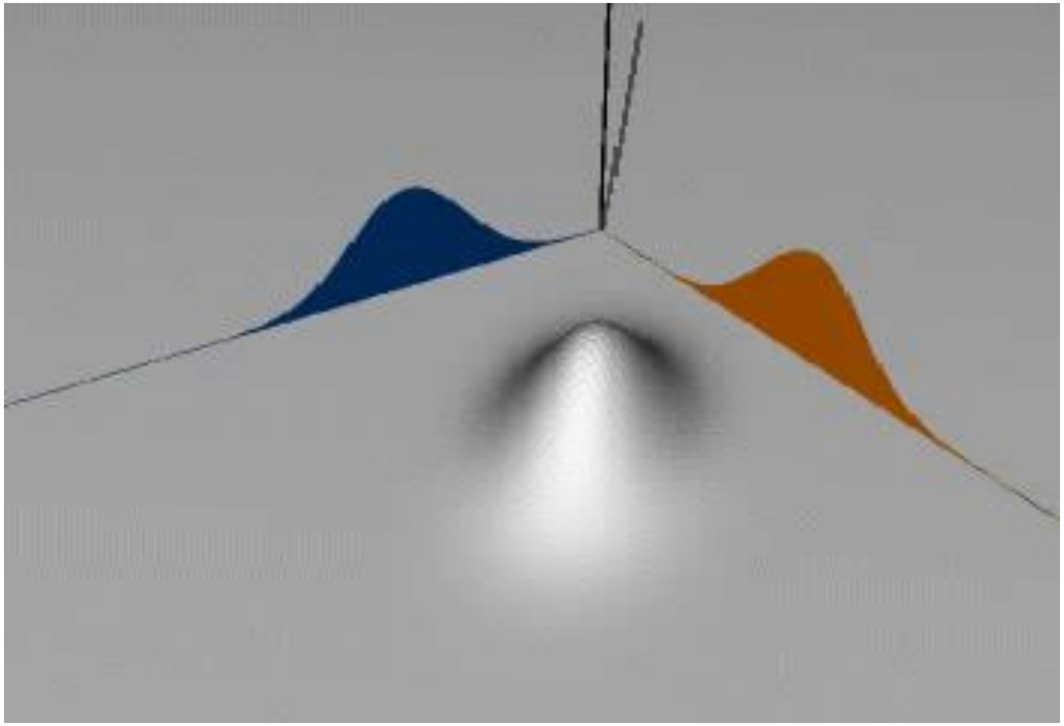
Hence $$f_{1|2}\left(x_1 \middle| x_2\right) = \frac{1}{\sqrt{2\pi}b}\ e^{-\frac{1}{2}\left(\frac{x_1 - a}{b}\right)^2}$$

Thus the conditional distribution of x_2 given x_1 is Normal with:

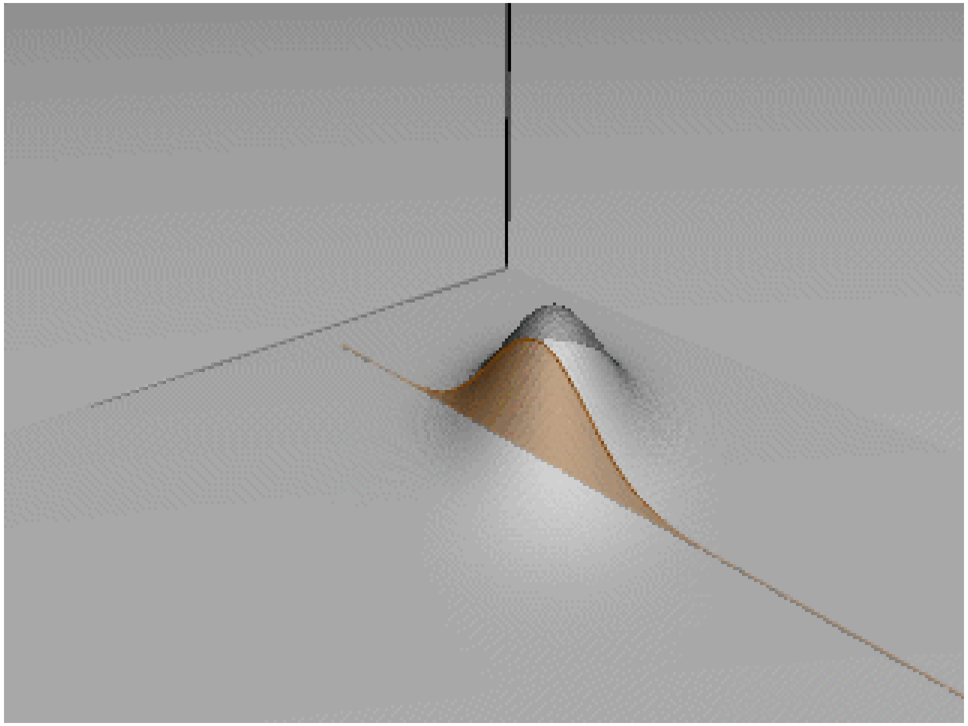
mean $a = \mu_{1|2} = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}\left(x_2 - \mu_2\right)$ and

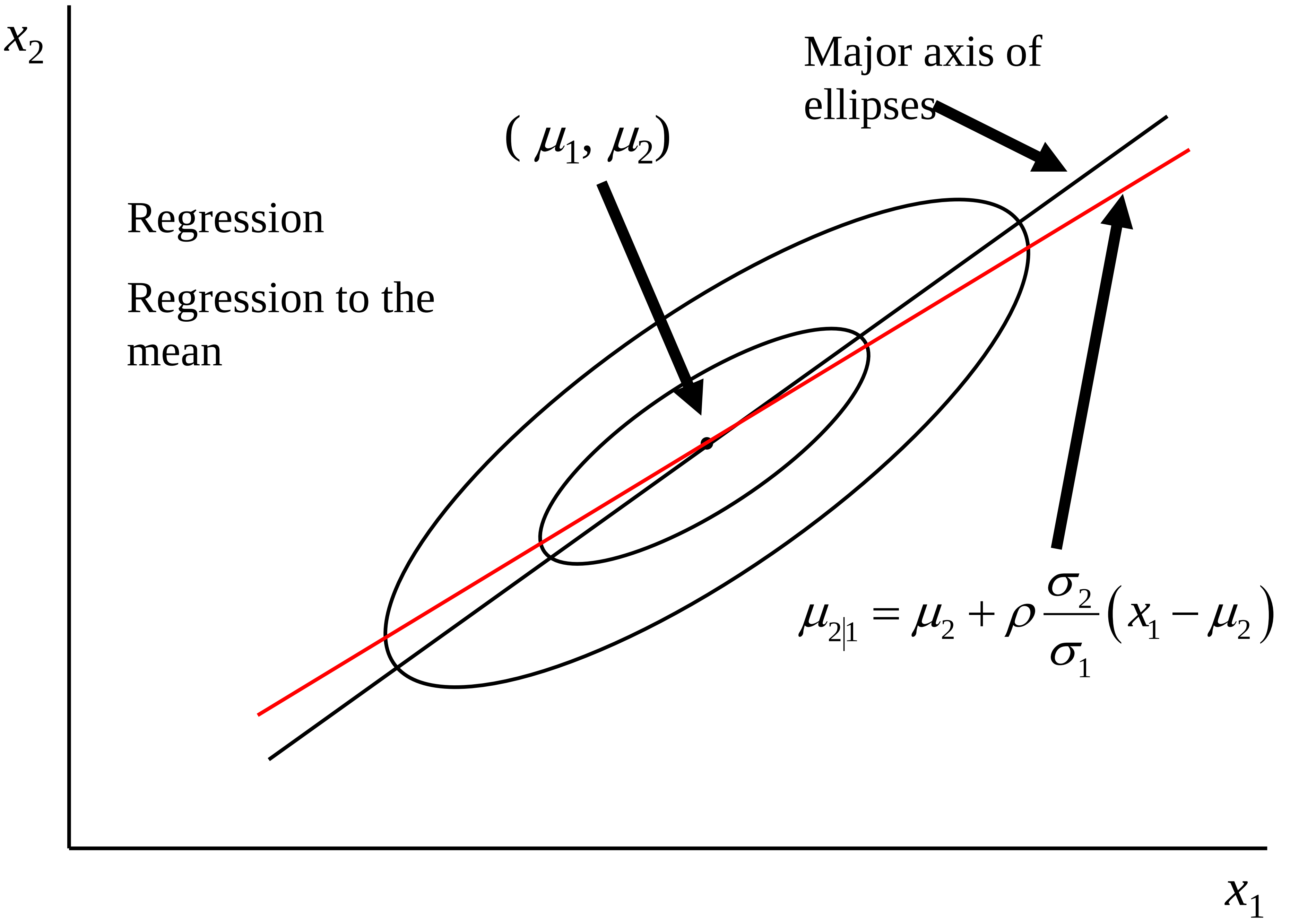
standard deviation $b = \sigma_{1|2} = \sigma_1 \sqrt{1 - \rho^2}$

Bivariate Normal Distribution with marginal distributions



Bivariate Normal Distribution with conditional distribution





x_2
Major axis of ellipses
(μ_1, μ_2)
Regression
Regression to the mean
$\mu_{2|1} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x_1 - \mu_2)$
x_1